

# MINIMAL PSEUDOCOMPACT GROUP TOPOLOGIES ON FREE ABELIAN GROUPS

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*Dedicated to Robert Lowen on the occasion of his 60th anniversary*

**ABSTRACT.** A Hausdorff topological group  $G$  is minimal if every continuous isomorphism  $f : G \rightarrow H$  between  $G$  and a Hausdorff topological group  $H$  is open. Significantly strengthening a 1981 result of Stoyanov, we prove the following theorem: For every infinite minimal abelian group  $G$  there exists a sequence  $\{\sigma_n : n \in \mathbb{N}\}$  of cardinals such that

$$w(G) = \sup\{\sigma_n : n \in \mathbb{N}\} \quad \text{and} \quad \sup\{2^{\sigma_n} : n \in \mathbb{N}\} \leq |G| \leq 2^{w(G)},$$

where  $w(G)$  is the weight of  $G$ . If  $G$  is an infinite minimal abelian group, then either  $|G| = 2^\sigma$  for some cardinal  $\sigma$ , or  $w(G) = \min\{\sigma : |G| \leq 2^\sigma\}$ ; moreover, the equality  $|G| = 2^{w(G)}$  holds whenever  $\text{cf}(w(G)) > \omega$ .

For a cardinal  $\kappa$ , we denote by  $F_\kappa$  the free abelian group with  $\kappa$  many generators. If  $F_\kappa$  admits a pseudocompact group topology, then  $\kappa \geq \mathfrak{c}$ , where  $\mathfrak{c}$  is the cardinality of the continuum. We show that the existence of a minimal pseudocompact group topology on  $F_\mathfrak{c}$  is equivalent to the Lusin's Hypothesis  $2^{\omega_1} = \mathfrak{c}$ . For  $\kappa > \mathfrak{c}$ , we prove that  $F_\kappa$  admits a (zero-dimensional) minimal pseudocompact group topology if and only if  $F_\kappa$  has both a minimal group topology and a pseudocompact group topology. If  $\kappa > \mathfrak{c}$ , then  $F_\kappa$  admits a connected minimal pseudocompact group topology of weight  $\sigma$  if and only if  $\kappa = 2^\sigma$ . Finally, we establish that no infinite torsion-free abelian group can be equipped with a locally connected minimal group topology.

*Throughout this paper all topological groups are Hausdorff.* We denote by  $\mathbb{Z}$ ,  $\mathbb{P}$  and  $\mathbb{N}$  respectively the set of integers, the set of primes and the set of natural numbers. Moreover  $\mathbb{Q}$  denotes the group of rationals and  $\mathbb{R}$  the group of reals. For  $p \in \mathbb{P}$  the symbol  $\mathbb{Z}_p$  is used for the group of  $p$ -adic integers. The symbol  $\mathfrak{c}$  stands for the cardinality of the continuum. For a topological group  $G$  the symbol  $w(G)$  stands for the weight of  $G$ . The Pontryagin dual of a topological abelian group  $G$  is denoted by  $\widehat{G}$ . If  $H$  is a group and  $\sigma$  is a cardinal, then  $H^{(\sigma)}$  is used to denote the direct sum of  $\sigma$  many copies of the group  $H$ . If  $G$  and  $H$  are groups, then a map  $f : G \rightarrow H$  is called a *monomorphism* provided that  $f$  is both a group homomorphism and an injection. For undefined terms see [16, 17].

**Definition 0.1.** For a cardinal  $\kappa$  we use  $F_\kappa$  to denote the free abelian group with  $\kappa$  many generators.

## 1. INTRODUCTION

The following notion was introduced independently by Choquet (see Doitchinov [14]) and Stephenson [24].

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**Definition 1.1.** A Hausdorff group topology  $\tau$  on a group  $G$  is called *minimal* provided that every Hausdorff group topology  $\tau'$  on  $G$  such that  $\tau' \subseteq \tau$  satisfies  $\tau' = \tau$ . Equivalently, a Hausdorff topological group  $G$  is minimal if every continuous isomorphism  $f : G \rightarrow H$  between  $G$  and a Hausdorff topological group  $H$  is a topological isomorphism.

There exist abelian groups which admit no minimal group topologies at all, e.g., the group of rational numbers  $\mathbb{Q}$  [21] or Prüfer's group  $\mathbb{Z}(p^\infty)$  [11]. This suggests the general problem to determine the algebraic structure of the minimal abelian groups, or equivalently, the following

**Problem 1.2.** [9, Problem 4.1] *Describe the abelian groups that admit minimal group topologies.*

Prodanov solved Problem 1.2 first for all free abelian groups of finite rank [20], and later on he improved this result extending it to all cardinals  $\leq \mathfrak{c}$  [21]:

**Theorem 1.3.** [20, 21] *For every cardinal  $\kappa \leq \mathfrak{c}$ , the group  $F_\kappa$  admits minimal group topologies.*

Since  $|F_\kappa| = \omega \cdot \kappa$  for each cardinal  $\kappa$ , uncountable free abelian groups are determined up to isomorphism by their cardinality. This suggests the problem of characterizing the cardinality of minimal abelian groups. The following set-theoretic definition is ultimately relevant to this problem.

**Definition 1.4.** (i) For infinite cardinals  $\kappa$  and  $\sigma$  the symbol  $\mathbf{Min}(\kappa, \sigma)$  denotes the following statement: There exists a sequence of cardinals  $\{\sigma_n : n \in \mathbb{N}\}$  such that

$$(1) \quad \sigma = \sup_{n \in \mathbb{N}} \sigma_n \quad \text{and} \quad \sup_{n \in \mathbb{N}} 2^{\sigma_n} \leq \kappa \leq 2^\sigma.$$

We say that the sequence  $\{\sigma_n : n \in \mathbb{N}\}$  as above *witnesses*  $\mathbf{Min}(\kappa, \sigma)$ .

- (ii) An infinite cardinal number  $\kappa$  satisfying  $\mathbf{Min}(\kappa, \sigma)$  for some infinite cardinal  $\sigma$  will be called a *Stoyanov* cardinal.
- (iii) For the sake of convenience, we add to the class of Stoyanov cardinals also all finite cardinals.

The cardinals from item (ii) in the above definition were first introduced by Stoyanov in [25] under the name “permissible cardinals”. Their importance is evident from the following fundamental result of Stoyanov providing a complete characterization of the possible cardinalities of minimal abelian groups, thereby solving Problem 1.2 for all free abelian groups:

**Theorem 1.5.** [25]

- (a) *If  $G$  is a minimal abelian group, then  $|G|$  is a Stoyanov cardinal.*
- (b) *For a cardinal  $\kappa$ ,  $F_\kappa$  admits minimal group topologies if and only if  $\kappa$  is a Stoyanov cardinal.*

If  $\kappa$  is a finite cardinal satisfying (1), then  $\kappa = 2^n$  for some  $n \in \mathbb{N}$ . On the other hand, every finite group is compact and thus minimal. Furthermore, the group  $F_n$  admits minimal group topologies for every  $n \in \mathbb{N}$  by Theorem 1.3. It is in order to include also the case of finite groups in Theorem 1.5(a) and finitely generated groups in Theorem 1.5(b) that we decided to add item (iii) to Definition 1.4.

It is worth noting that the commutativity of the group in Theorem 1.5(b) is important because all restrictions on the cardinality disappear in the case of (non-abelian) free groups:

**Theorem 1.6.** [23, 22] *Every free group admits a minimal group topology.*

For free groups with infinitely many generators this theorem has been proved in [23]. The remaining case was covered in [22].

A subgroup  $H$  of a topological group  $G$  is *essential* (in  $G$ ) if  $H \cap N \neq \{e\}$  for every closed normal subgroup  $N$  of  $G$  with  $N \neq \{e\}$ , where  $e$  is the identity element of  $G$  [20, 24]. This notion is a crucial ingredient of the so-called “minimality criterion”, due to Prodanov and Stephenson [20, 24], describing the dense minimal subgroups of compact groups.

**Theorem 1.7.** ([20, 24]; see also [10, 12]) *A dense subgroup  $H$  of a compact group  $G$  is minimal if and only if  $H$  is essential in  $G$ .*

A topological group  $G$  is *pseudocompact* if every continuous real-valued function defined on  $G$  is bounded [18]. In the spirit of Theorem 1.5(b) characterizing the free abelian groups admitting *minimal* topologies, one can also describe the free abelian groups that admit *pseudocompact* group topologies ([5, 13]; see Theorem 4.4). The aim of this article is to provide *simultaneous* minimal and pseudocompact topologization of free abelian groups. To achieve this goal, we need an alternative description of Stoyanov cardinals obtained in Proposition 3.5 as well as a more precise form of Theorem 1.5(a) given in Theorem 2.1.

We finish this section with a fundamental restriction on the size of pseudocompact groups due to van Douwen.

**Theorem 1.8.** [26] *If  $G$  is an infinite pseudocompact group, then  $|G| \geq \mathfrak{c}$ .*

## 2. MAIN RESULTS

**2.1. Cardinality and weight of minimal abelian groups.** Let  $\kappa$  be a cardinal. Recall that the *cofinality*  $\text{cf}(\kappa)$  of  $\kappa$  is defined to be the smallest cardinal  $\varkappa$  such that there exists a transfinite sequence  $\{\tau_\alpha : \alpha \in \varkappa\}$  of cardinals such that  $\kappa = \sup\{\tau_\alpha : \alpha \in \varkappa\}$  and  $\tau_\alpha < \kappa$  for all  $\alpha \in \varkappa$ . We say that  $\kappa$  is *exponential* if  $\kappa = 2^\sigma$  for some cardinal  $\sigma$ , and we call  $\kappa$  *non-exponential* otherwise. Recall that  $\kappa$  is called a *strong limit* provided that  $2^\mu < \kappa$  for every cardinal  $\mu < \kappa$ . When  $\kappa$  is infinite, we define  $\log \kappa = \min\{\sigma : \kappa \leq 2^\sigma\}$ .

We start this section with a much sharper version of Theorem 1.5(a) showing that the weight  $w(G)$  of a minimal abelian group  $G$  can be taken as the cardinal  $\sigma$  from Definition 1.4(ii) witnessing that  $|G|$  is a Stoyanov cardinal:

**Theorem 2.1.** *If  $G$  is an infinite minimal abelian group, then  $\text{Min}(|G|, w(G))$  holds.*

This theorem, along with the complete “internal” characterization of the Stoyanov cardinals obtained in Proposition 3.5 permits us to establish some new important relations between the cardinality and the weight of an arbitrary minimal abelian group.

**Theorem 2.2.** *If  $\kappa$  is a cardinal with  $\text{cf}(\kappa) > \omega$  and  $G$  is a minimal abelian group such that  $w(G) \geq \kappa$ , then  $|G| \geq 2^\kappa$ .*

Let us recall that  $|G| = 2^{w(G)}$  holds for every compact group  $G$  [3]. Taking  $\kappa = w(G)$  in Theorem 2.2 we obtain the following extension of this property to all minimal abelian groups:

**Corollary 2.3.** *Let  $G$  be a minimal abelian group with  $\text{cf}(w(G)) > \omega$ . Then  $|G| = 2^{w(G)}$ .*

Example 8.3(a) below and Theorem 1.6 show that neither  $\text{cf}(w(G)) > \omega$  nor “abelian” can be removed in Corollary 2.3.

Taking  $\kappa = \omega_1$  in Theorem 2.2 one obtains the following surprising metrizability criterion for “small” minimal abelian groups:

**Corollary 2.4.** *A minimal abelian group of size  $< 2^{\omega_1}$  is metrizable.*

The condition  $\text{cf}(w(G)) > \omega$  plays a prominent role in the above results. In particular, Corollary 2.3 implies that  $\text{cf}(w(G)) = \omega$  for a minimal abelian group with  $|G| < 2^{w(G)}$ . Our next theorem gives a more precise information in this direction.

**Theorem 2.5.** *Let  $G$  be an infinite minimal abelian group such that  $|G|$  is a non-exponential cardinal. Then  $w(G) = \log |G|$  and  $\text{cf}(w(G)) = \omega$ .*

Under the assumption of GCH, the equality  $w(G) = \log |G|$  holds true for every compact group. Theorem 2.5 establishes this property in ZFC for all minimal abelian groups of non-exponential size. Let us note that the restraint “non-exponential” cannot be omitted, even in the compact case. Indeed, the equality  $w(G) = \log |G|$  may fail for compact abelian groups: Under the Lusin’s Hypothesis  $2^{\omega_1} = \mathfrak{c}$ , for the group  $G = \mathbb{Z}(2)^{\omega_1}$  one has  $w(G) = \omega_1 \neq \omega = \log \mathfrak{c} = \log |G|$ .

**Example 2.6.** There exists a consistent example of a compact abelian group  $G$  such that  $\text{cf}(w(G)) = \omega$  and  $w(G) > \log |G|$  (see Example 3.4 (b)).

**2.2. Minimal pseudocompact group topologies on free abelian groups.** Since pseudocompact metric spaces are compact, from Corollary 2.4 we immediately get the following:

**Corollary 2.7.** *Let  $G$  be an abelian group such that  $|G| < 2^{\omega_1}$ . Then  $G$  admits a minimal pseudocompact group topology if and only if  $G$  admits a compact metric group topology.*

By Theorem 1.8, this corollary is vacuously true under the Lusin’s Hypothesis  $2^{\omega_1} = \mathfrak{c}$ .

Corollary 2.7 shows that for abelian groups of “small size” minimal and pseudocompact topologizations are connected in some sense by compactness. We shall see in Corollary 8.2 below that the same phenomenon happens for divisible abelian groups, irrespectively of their size.

Rather surprisingly, the mere existence of a minimal group topology on  $F_\kappa$  quite often implies the existence of a group topology on  $F_\kappa$  that is both minimal and pseudocompact. In other words, one often gets pseudocompactness “for free”.

**Theorem 2.8.** *Let  $\kappa$  and  $\sigma$  be infinite cardinals. Assume also that  $\sigma$  is not a strong limit. If  $F_\kappa$  admits a minimal group topology of weight  $\sigma$ , then  $F_\kappa$  also admits a zero-dimensional minimal pseudocompact group topology of weight  $\sigma$ .*

Recall that the *beth cardinals*  $\beth_\alpha$  are defined by recursion on  $\alpha$  as follows. Let  $\beth_0 = \omega$ . If  $\alpha = \beta + 1$  is a successor ordinal, then  $\beth_\alpha = 2^{\beth_\beta}$ . If  $\alpha$  is a limit ordinal, then  $\beth_\alpha = \sup\{\beth_\beta : \beta \in \alpha\}$ .

The restriction on weight in Theorem 2.8 is necessary, as our next example demonstrates.

**Example 2.9.** Let  $\kappa = \beth_\omega$ . Clearly, the sequence  $\{\beth_n : n \in \mathbb{N}\}$  witnesses that  $\kappa$  is a Stoyanov cardinal, so  $F_\kappa$  admits a minimal group topology  $\tau$  by Theorem 1.5(b). On the other hand, since  $\kappa$  is a strong limit cardinal with  $\text{cf}(\kappa) = \omega$  and  $|F_\kappa| = \kappa$ , the group  $F_\kappa$  does not admit any pseudocompact group topology by the result of van Douwen [26]. Note that  $w(F_\kappa, \tau) = \log |F_\kappa| = \log \kappa = \kappa$  by Theorem 2.5, so  $\sigma = w(F_\kappa, \tau)$  is a strong limit cardinal.

“Going in the opposite direction”, in Example 4.7 below we will define a cardinal  $\kappa$  such that  $F_\kappa$  admits a pseudocompact group topology of weight  $\sigma$  that is not a strong limit cardinal, and yet  $F_\kappa$  does not admit any minimal group topology. These two examples show that the existence of a minimal group topology and the existence of a pseudocompact group topology on a free abelian group are “independent events”.

For a free group of size  $> \mathfrak{c}$  that admits both a minimal group topology and a pseudocompact group topology, the next theorem discovers the surprising possibility of “simultaneous

topologization" with a topology which is both minimal and pseudocompact. Moreover, it turns out that this topology can also be chosen to be zero-dimensional.

**Theorem 2.10.** *For every cardinal  $\kappa > \mathfrak{c}$  the following conditions are equivalent:*

- (a)  $F_\kappa$  admits both a minimal group topology and a pseudocompact group topology;
- (b)  $F_\kappa$  admits a minimal pseudocompact group topology;
- (c)  $F_\kappa$  admits a zero-dimensional minimal pseudocompact group topology.

The free abelian group  $F_\mathfrak{c}$  of cardinality  $\mathfrak{c}$  admits a minimal group topology (Theorem 1.3) and a pseudocompact group topology [13]. Our next theorem shows that the statement " $F_\mathfrak{c}$  admits a minimal pseudocompact group topology" is both consistent with and independent of ZFC.

**Theorem 2.11.** *The following conditions are equivalent:*

- (a)  $F_\mathfrak{c}$  admits a minimal pseudocompact group topology;
- (b)  $F_\mathfrak{c}$  admits a connected minimal pseudocompact group topology;
- (c)  $F_\mathfrak{c}$  admits a zero-dimensional minimal pseudocompact group topology;
- (d) the Lusin's Hypothesis  $2^{\omega_1} = \mathfrak{c}$  holds.

Since every infinite pseudocompact group has cardinality  $\geq \mathfrak{c}$  (Theorem 1.8), Theorems 2.10 and 2.11 provide a complete description of free abelian groups that have a minimal (zero-dimensional) pseudocompact group topology. The equivalence of (a) and (b) in Theorem 2.10 (respectively, (a) and (d) in Theorem 2.11) was announced without proof in [9, Theorem 4.11].

Motivated by Theorem 2.10(c) and Theorem 2.11(c), where the minimal pseudocompact topology can be additionally chosen zero-dimensional (or connected, in Theorem 2.11(b)), we arrive at the following natural question: *If  $\kappa$  is a cardinal such that  $F_\kappa$  admits a minimal group topology  $\tau_1$  and a pseudocompact group topology  $\tau_2$ , and one of these topologies is connected, does then  $F_\kappa$  admit a connected minimal pseudocompact group topology  $\tau_3$ ?* Theorem 2.11 answers this question in the case of  $F_\mathfrak{c}$ . The next theorem gives an answer for  $\kappa > \mathfrak{c}$ , showing a symmetric behavior, as far as connectedness is concerned. This should be compared with the equivalent items in Theorem 2.11 where item (a) contains no restriction beyond minimality and pseudocompactness, whereas item (c) contains "zero-dimensional".

**Theorem 2.12.** *Let  $\kappa$  and  $\sigma$  be infinite cardinals with  $\kappa > \mathfrak{c}$ . The following conditions are equivalent:*

- (a)  $F_\kappa$  admits a connected minimal pseudocompact group topology (of weight  $\sigma$ );
- (b)  $F_\kappa$  admits a connected minimal group topology (of weight  $\sigma$ );
- (c)  $\kappa$  is exponential ( $\kappa = 2^\sigma$ ).

This theorem is "asymmetric" in some sense toward minimality. Indeed, item (b) should be compared with the fact that the existence of a connected pseudocompact group topology on  $F_\kappa$  need not necessarily imply that  $F_\kappa$  admits a connected minimal group topology (see Example 4.8).

If a free abelian group admits a pseudocompact group topology, then it admits also a pseudocompact group topology which is both connected and locally connected [13, Theorem 5.10]. When minimality is added to the mix, the situation becomes totally different. In Example 4.8 below we exhibit a free abelian group  $F_\kappa$  that admits a connected, locally connected, pseudocompact group topology, and yet  $F_\kappa$  does not have any connected minimal group topology. Even more striking is the following

**Theorem 2.13.** *A locally connected minimal torsion-free abelian group is trivial.*

Theorem 2.13 strengthens significantly [13, Corollary 8.8] by replacing “compact” in it with “minimal”.

**Corollary 2.14.** *No free abelian group admits a locally connected, minimal group topology.*

The reader may wish to compare this corollary with Theorems 2.11 and 2.12.

The paper is organized as follows. In Section 3 we give some properties of Stoyanov cardinals, while Section 4 contains all necessary facts concerning pseudocompact topologization. The culmination here is Corollary 4.12 establishing that, roughly speaking, if  $F_\kappa$  admits a minimal group topology  $\tau_1$  and a pseudocompact group topology  $\tau_2$ , then one can assume, without loss of generality, that this pair satisfies  $w(F_\kappa, \tau_1) = w(F_\kappa, \tau_2)$ . Sections 5 and 6 prepare the remaining necessary tools for the proof of the main results, deferred to Section 7. Finally, in Section 8 we discuss the counterpart of the simultaneous minimal and pseudocompact topologization for other classes of abelian groups such as divisible groups, torsion-free groups and torsion groups, as well as the same problem for (non-commutative) free groups.

### 3. PROPERTIES OF STOYANOV CARDINALS

We start with an example of small Stoyanov cardinals.

**Example 3.1.** If  $\omega \leq \kappa \leq \mathfrak{c}$ , then  $\mathbf{Min}(\kappa, \omega)$ .

In our next example we discuss the connection between  $\mathbf{Min}(\kappa, \sigma)$  and the property of  $\kappa$  to be exponential.

**Example 3.2.** Let  $\kappa$  be an infinite cardinal.

- (a) *If  $\kappa = 2^\sigma$ , then  $\mathbf{Min}(\kappa, \sigma)$  holds. In particular, an exponential cardinal is Stoyanov.*
- (b) *If  $\{\sigma_n : n \in \mathbb{N}\}$  is a sequence of cardinals witnessing  $\mathbf{Min}(\kappa, \sigma)$  such that  $\sigma = \sigma_m$  for some  $m \in \mathbb{N}$ , then  $\kappa = 2^\sigma$ .* Indeed, (1) and our assumption yield

$$2^\sigma = 2^{\sigma_m} \leq \sup_{n \in \mathbb{N}} 2^{\sigma_n} \leq \kappa \leq 2^\sigma.$$

Hence  $\kappa = 2^\sigma$ .

- (c) *If  $\text{cf}(\sigma) > \omega$ , then  $\mathbf{Min}(\kappa, \sigma)$  if and only if  $\kappa = 2^\sigma$ .* If  $\kappa = 2^\sigma$ , then  $\mathbf{Min}(\kappa, \sigma)$  by item (a). Assume  $\mathbf{Min}(\kappa, \sigma)$ , and let  $\{\sigma_n : n \in \mathbb{N}\}$  be a sequence of cardinals witnessing  $\mathbf{Min}(\kappa, \sigma)$ . From (1) and  $\text{cf}(\sigma) > \omega$  we get  $\sigma = \sigma_m$  for some  $m \in \mathbb{N}$ . Applying item (b) gives  $\kappa = 2^\sigma$ .

Clearly,  $\mathbf{Min}(\kappa, \sigma)$  implies  $\sigma \geq \log \kappa$ . We show now that this inequality becomes an equality in case  $\kappa$  is non-exponential.

**Lemma 3.3.** *Let  $\kappa$  be a non-exponential infinite cardinal. Then:*

- (a)  $\mathbf{Min}(\kappa, \sigma)$  if and only if  $\text{cf}(\sigma) = \omega$  and  $\log \kappa = \sigma$ ;
- (b)  $\mathbf{Min}(\kappa, \log \kappa)$  if and only if  $\text{cf}(\log \kappa) = \omega$ .

*Proof.* (a) To prove the “only if” part, assume that  $\mathbf{Min}(\kappa, \sigma)$  holds, and let  $\{\sigma_n : n \in \mathbb{N}\}$  be a sequence of cardinals witnessing  $\mathbf{Min}(\kappa, \sigma)$ . Since  $\kappa \leq 2^\sigma$  by (1), we have  $\log \kappa \leq \sigma$ . Assume  $\log \kappa < \sigma$ . From (1) we conclude that  $\log \kappa \leq \sigma_m$  for some  $m \in \mathbb{N}$ . Therefore

$$2^{\log \kappa} \leq 2^{\sigma_m} \leq \sup_{n \in \mathbb{N}} 2^{\sigma_n} \leq \kappa \leq 2^{\log \kappa}$$

by (1). Thus  $\kappa = 2^{\log \kappa}$  is an exponential cardinal, a contradiction. This proves that  $\sigma = \log \kappa$ .

To prove the “if” part, assume that  $\text{cf}(\sigma) = \omega$  and  $\log \kappa = \sigma$ . Then there exists a sequence of cardinals  $\{\sigma_n : n \in \mathbb{N}\}$  such that  $\sigma = \sup_{n \in \mathbb{N}} \sigma_n$  and  $\sigma_n < \sigma = \log \kappa$  for every  $n \in \mathbb{N}$ . In particular,  $2^{\sigma_n} < \kappa$  for every  $n \in \mathbb{N}$ . Consequently,

$$\sup_{n \in \mathbb{N}} 2^{\sigma_n} \leq \kappa \leq 2^{\log \kappa} = 2^\sigma.$$

That is, (1) holds. Therefore, the sequence  $\{\sigma_n : n \in \mathbb{N}\}$  witnesses  $\text{Min}(\kappa, \sigma)$ .

Item (b) follows from item (a).  $\square$

**Example 3.4.** Let  $\kappa$  and  $\sigma$  be cardinals. According to Example 3.2(a),  $\text{Min}(\kappa, \sigma)$  does not imply  $\text{cf}(\sigma) = \omega$  in case  $\kappa$  is exponential. (Indeed, it suffices to take  $\kappa = 2^\sigma$  with  $\text{cf}(\sigma) > \omega$ .)

(a) Let us show that the assumption “ $\kappa$  is non-exponential” in Lemma 3.3(a) is necessary (to prove that  $\text{Min}(\kappa, \sigma)$  implies  $\log \kappa = \sigma$ ) even in the case  $\text{cf}(\sigma) = \omega$ . To this end, use an appropriate Easton model [15] satisfying

$$2^{\omega_{\omega+1}} = \omega_{\omega+2} \quad \text{and} \quad 2^{\omega_n} = \omega_{\omega+2} \text{ for all } n \in \mathbb{N}.$$

Let  $\kappa = \omega_{\omega+2}$  and  $\sigma = \omega_\omega$ . Then  $2^\sigma = \kappa$  as  $2^{\omega_{\omega+1}} = 2^{\omega_n} = \kappa$  for every  $n \in \mathbb{N}$ . So  $\text{Min}(\kappa, \sigma)$  holds by Example 3.2(a). Moreover  $\text{cf}(\sigma) = \omega$  and  $\log \kappa = \omega_0 < \omega_\omega = \sigma$ .

(b) Using the cardinals  $\kappa$  and  $\sigma$  from item (a) we can give now the example anticipated in Example 2.6. Let  $G = \mathbb{Z}(2)^\sigma$ . Then  $w(G) = \sigma$ , so  $\text{cf}(w(G)) = \omega$  and yet  $\log |G| = \log 2^\sigma = \log \kappa = \omega_0 < \sigma = w(G)$ .

The next proposition, summarizing the above results, provides an alternative description of the infinite Stoyanov cardinals that makes no use of the somewhat “external” condition (1).

**Proposition 3.5.** *Let  $\kappa$  be an infinite cardinal.*

(a) *If  $\kappa$  is exponential, then  $\text{Min}(\kappa, \sigma)$  holds for every cardinal  $\sigma$  with  $\kappa = 2^\sigma$ .*  
 (b) *If  $\kappa$  is non-exponential, then  $\text{Min}(\kappa, \sigma)$  is equivalent to  $\sigma = \log \kappa$  and  $\text{cf}(\log \kappa) = \omega$ .*

*Proof.* Item (a) follows from Example 3.2(a), and item (b) follows from Lemma 3.3(a).  $\square$

#### 4. CARDINAL INVARIANTS RELATED TO PSEUDOCOMPACT GROUPS

Recall that a subset  $Y$  of a space  $X$  is said to be  $G_\delta$ -dense in  $X$  provided that  $Y \cap B \neq \emptyset$  for every non-empty  $G_\delta$ -subset  $B$  of  $X$ .

The following theorem describes pseudocompact groups in terms of their completion.

**Theorem 4.1.** [7, Theorem 4.1] *A precompact group  $G$  is pseudocompact if and only if  $G$  is  $G_\delta$ -dense in its completion.*

**Definition 4.2.** (i) If  $X$  is a non-empty set and  $\sigma$  is an infinite cardinal, then a set  $F \subseteq X^\sigma$  is  $\omega$ -dense in  $X^\sigma$ , provided that for every countable set  $A \subseteq \sigma$  and each function  $\varphi \in X^A$  there exists  $f \in F$  such that  $f(\alpha) = \varphi(\alpha)$  for all  $\alpha \in A$ .  
 (ii) If  $\kappa$  and  $\sigma \geq \omega$  are cardinals, then  $\text{Ps}(\kappa, \sigma)$  abbreviates the sentence “there exists an  $\omega$ -dense set  $F \subseteq \{0, 1\}^\sigma$  with  $|F| = \kappa$ ”.  
 (iii) For an infinite cardinal  $\sigma$  let  $m(\sigma)$  denote the minimal cardinal  $\kappa$  such that  $\text{Ps}(\kappa, \sigma)$  holds.

Items (i) and (ii) of the above definition are taken from [2] except for the notation  $\text{Ps}(\kappa, \sigma)$  that appears in [13, Definition 2.6]. Item (iii) is equivalent to the definition of the cardinal function  $m(\sigma)$  of Comfort and Robertson [4]. It is worth noting that  $m(\sigma) = \delta(\sigma)$  for every infinite cardinal  $\sigma$ , where  $\delta(-)$  is the cardinal function defined by Cater, Erdős and Galvin [2].

The set-theoretical condition  $\mathbf{Ps}(\kappa, \sigma)$  is ultimately related to the existence of pseudocompact group topologies.

**Theorem 4.3.** ([4]; see also [13, Fact 2.12 and Theorem 3.3(i)]) *Let  $\kappa$  and  $\sigma \geq \omega$  be cardinals. Then  $\mathbf{Ps}(\kappa, \sigma)$  holds if and only if there exists a group  $G$  of cardinality  $\kappa$  which admits a pseudocompact group topology of weight  $\sigma$ .*

Moreover, the condition  $\mathbf{Ps}(\kappa, \sigma)$  completely describes free abelian groups that admit pseudocompact group topologies.

**Theorem 4.4.** ([5], [13, Theorem 5.10]) *If  $\kappa$  is a cardinal, then  $F_\kappa$  admits a pseudocompact group topology of weight  $\sigma$  if and only if  $\mathbf{Ps}(\kappa, \sigma)$  holds.*

In the next lemma we summarize some properties of the cardinal function  $m(-)$  for future reference.

**Lemma 4.5.** ([2]; see also [4, Theorem 2.7]) *Let  $\sigma$  be an infinite cardinal. Then:*

- (a)  $m(\sigma) \geq 2^\omega$  and  $\text{cf}(m(\sigma)) > \omega$ ;
- (b)  $\log \sigma \leq m(\sigma) \leq (\log \sigma)^\omega$ ;
- (c)  $m(\lambda) \leq m(\sigma)$  whenever  $\lambda$  is a cardinal with  $\lambda \leq \sigma$ .

Some useful properties of the condition  $\mathbf{Ps}(\lambda, \kappa)$  are collected in the next proposition. Items (a) and (b) are part of [13, Lemmas 2.7 and 2.8], and items (d) and (e) are particular cases of [13, Lemma 3.4(i)].

**Proposition 4.6.** (a)  $\mathbf{Ps}(\mathfrak{c}, \omega)$  holds, and moreover,  $m(\omega) = \mathfrak{c}$ ; also  $\mathbf{Ps}(\mathfrak{c}, \omega_1)$  holds.  
 (b) If  $\mathbf{Ps}(\kappa, \sigma)$  holds for some cardinals  $\kappa$  and  $\sigma \geq \omega$ , then  $\kappa \geq \mathfrak{c}$ , and  $\mathbf{Ps}(\kappa', \sigma)$  holds for every cardinal  $\kappa'$  such that  $\kappa \leq \kappa' \leq 2^\sigma$ .  
 (c) For cardinals  $\kappa$  and  $\sigma \geq \omega$ ,  $\mathbf{Ps}(\kappa, \sigma)$  holds if and only if  $m(\sigma) \leq \kappa \leq 2^\sigma$ .  
 (d)  $\mathbf{Ps}(2^\sigma, \sigma)$  and  $\mathbf{Ps}(2^\sigma, 2^{2^\sigma})$  hold for every infinite cardinal  $\sigma$ .  
 (e) If  $\sigma$  is a cardinal such that  $\sigma^\omega = \sigma$ , then  $\mathbf{Ps}(\sigma, 2^\sigma)$  holds.

**Example 4.7.** Let  $\kappa = \beth_{\omega_1}$  (see the text preceding Example 2.9 for the definition of  $\beth_{\omega_1}$ ). One can easily see that  $\kappa$  is not a Stoyanov cardinal (this was first noted by Stoyanov himself). Therefore, the group  $F_\kappa$  does not admit any minimal group topology by Theorem 1.5(a). On the other hand,  $\kappa = \kappa^\omega$  and Proposition 4.6(e) yield that  $\mathbf{Ps}(\kappa, 2^\kappa)$  holds. Applying Theorem 4.4 we conclude that  $F_\kappa$  admits a pseudocompact group topology of weight  $2^\kappa$ . In particular,  $\sigma = 2^\kappa$  is not a strong limit.

Example 4.7 should be compared with Theorem 2.8 where we show that if  $F_\kappa$  admits a minimal group topology of weight  $\sigma$  and  $\sigma$  is not a strong limit, then  $F_\kappa$  admits also a pseudocompact group topology of weight  $\sigma$ .

**Example 4.8.** Let  $\kappa$  be a non-exponential cardinal with  $\kappa = \kappa^\omega$  (e.g., a strong limit cardinal of uncountable cofinality). Then, according to Proposition 4.6(e),  $\mathbf{Ps}(\kappa, 2^\kappa)$  holds. Therefore  $F_\kappa$  admits a pseudocompact group topology (of weight  $2^\kappa$ ) that is both connected and locally connected [13, Theorem 5.10]. By Theorem 2.12,  $F_\kappa$  does not admit a connected minimal group topology as  $\kappa$  is non-exponential.

**Lemma 4.9.** *If  $\kappa$  and  $\sigma$  are infinite cardinals such that  $\sigma$  is not a strong limit cardinal, then  $\mathbf{Min}(\kappa, \sigma)$  implies  $\mathbf{Ps}(\kappa, \sigma)$ .*

*Proof.* Assume that  $\mathbf{Min}(\kappa, \sigma)$  holds, and let  $\{\sigma_n : n \in \mathbb{N}\}$  be a sequence of cardinals witnessing  $\mathbf{Min}(\kappa, \sigma)$ . Since  $\sigma$  is not a strong limit cardinal, there exists a cardinal  $\mu < \sigma$  such that  $\sigma \leq 2^\mu$ . Since  $\sigma = \sup_{n \in \mathbb{N}} \sigma_n$  by (1),  $\mu \leq \sigma_n$  for some  $n \in \mathbb{N}$ . Then  $\sigma \leq 2^\mu \leq 2^{\sigma_n}$ , and so  $\log \sigma \leq \sigma_n$ . Applying Lemma 4.5(b) and (1), we obtain

$$m(\sigma) \leq (\log \sigma)^\omega \leq \sigma_n^\omega \leq 2^{\sigma_n} \leq \kappa \leq 2^\sigma.$$

Hence  $\mathbf{Ps}(\kappa, \sigma)$  holds by Proposition 4.6(c).  $\square$

Our next example demonstrates that the restriction on the cardinal  $\sigma$  in Lemma 4.9 is necessary.

**Example 4.10.** Let  $\kappa$  be the Stoyanov cardinal from Example 2.9. From calculations in that example one concludes that  $\mathbf{Min}(\kappa, \kappa)$  holds. As was shown in Example 2.9,  $F_\kappa$  does not admit any pseudocompact group topology. Therefore,  $\mathbf{Ps}(\kappa, \sigma)$  fails for every cardinal  $\sigma$  (Theorem 4.4).

In the next lemma we show that, if  $\kappa$  is a Stoyanov cardinal satisfying  $\mathbf{Ps}(\kappa, \lambda)$  for some  $\lambda$ , then  $\mathbf{Ps}(\kappa, \sigma)$  holds also for the cardinal  $\sigma$  witnessing that  $\kappa$  is Stoyanov.

**Lemma 4.11.** *Let  $\kappa$  and  $\sigma$  be infinite cardinals satisfying  $\mathbf{Min}(\kappa, \sigma)$ . If  $\mathbf{Ps}(\kappa, \lambda)$  holds for some infinite cardinal  $\lambda$ , then  $\mathbf{Ps}(\kappa, \sigma)$  holds as well.*

*Proof.* By Lemma 4.9, it suffices only to consider the case when  $\sigma$  is a strong limit cardinal. Let  $\{\sigma_n : n \in \mathbb{N}\}$  be a sequence of cardinals witnessing  $\mathbf{Min}(\kappa, \sigma)$ . If  $\sigma = \sigma_n$  for some  $n \in \mathbb{N}$ , then  $\kappa = 2^\sigma$  by Example 3.2(b). Since  $\mathbf{Ps}(2^\sigma, \sigma)$  holds by Proposition 4.6(d), we are done in this case. Suppose now that  $\sigma > \sigma_n$  for every  $n \in \mathbb{N}$ . Since  $\mathbf{Ps}(\kappa, \lambda)$  holds, from Proposition 4.6(c) we get  $m(\lambda) \leq \kappa \leq 2^\lambda$ . If  $\lambda < \sigma$ , then  $2^\lambda < \sigma$  and so  $\kappa < \sigma$ . From (1) we get  $\kappa < \sigma_n$  for some  $n \in \mathbb{N}$ , and then  $\sigma_n < 2^{\sigma_n} \leq \kappa$ , a contradiction. Hence  $\sigma \leq \lambda$ . By Lemma 4.5(c)  $m(\sigma) \leq m(\lambda) \leq \kappa$ . Moreover  $\kappa \leq 2^\sigma$  by (1). It now follows from Proposition 4.6(c) that  $\mathbf{Ps}(\kappa, \sigma)$  holds.  $\square$

**Corollary 4.12.** *Let  $\kappa$  be a non-zero cardinal. If  $F_\kappa$  admits a minimal group topology  $\tau_1$  and a pseudocompact group topology  $\tau_2$ , then  $F_\kappa$  admits also a pseudocompact group topology  $\tau_3$  with  $w(F_\kappa, \tau_1) = w(F_\kappa, \tau_3)$ .*

*Proof.* From Theorem 1.8 we get  $\kappa \geq \mathfrak{c}$ . Define  $\sigma = w(F_\kappa, \tau_1)$ . Clearly,  $\sigma$  is infinite. Applying Theorem 2.1, we conclude that  $\mathbf{Min}(\kappa, \sigma)$  holds. Theorem 4.4 yields that  $\mathbf{Ps}(\kappa, \lambda)$  holds, where  $\lambda = w(F_\kappa, \tau_2)$ . Clearly,  $\lambda$  is infinite. Then  $\mathbf{Ps}(\kappa, \sigma)$  holds by Lemma 4.11. Finally, applying Theorem 4.4 once again, we obtain that  $F_\kappa$  must admit a pseudocompact group topology  $\tau_3$  such that  $w(F_\kappa, \tau_3) = \sigma$ .  $\square$

The proof of Corollary 4.12 relies on Theorem 2.1, which is proved later in Section 7. Nevertheless, this does not create any problems, because Corollary 4.12 is never used thereafter.

## 5. BUILDING $G_\delta$ -DENSE $\mathcal{V}$ -INDEPENDENT SUBSETS IN PRODUCTS

A *variety of groups*  $\mathcal{V}$  is a class of abstract groups closed under subgroups, quotients and products. For a variety  $\mathcal{V}$  and  $G \in \mathcal{V}$  a subset  $X$  of  $G$  is  $\mathcal{V}$ -independent if the subgroup  $\langle X \rangle$  of  $G$  generated by  $X$  belongs to  $\mathcal{V}$  and for each map  $f : X \rightarrow H \in \mathcal{V}$  there exists a unique homomorphism  $\bar{f} : \langle X \rangle \rightarrow H$  extending  $f$ . Moreover, the  $\mathcal{V}$ -rank of  $G$  is

$$r_{\mathcal{V}}(G) := \sup\{|X| : X \text{ is a } \mathcal{V}\text{-independent subset of } G\}.$$

In particular, if  $\mathcal{A}$  is the variety of all abelian groups, then the  $\mathcal{A}$ -rank is the usual free rank  $r(-)$ , and for the variety  $\mathcal{A}_p$  of all abelian groups of exponent  $p$  (for a prime  $p$ ) the  $\mathcal{A}_p$ -rank is the usual  $p$ -rank  $r_p(-)$ .

Our first lemma is a generalization of [13, Lemma 4.1] that is in fact equivalent to [13, Lemma 4.1] (as can be seen from its proof below).

**Lemma 5.1.** *Let  $\mathcal{V}$  be a variety of groups and  $I$  an infinite set. For every  $i \in I$  let  $H_i$  be a group such that  $r_{\mathcal{V}}(H_i) \geq \omega$ . Then  $r_{\mathcal{V}}(\prod_{i \in I} H_i) \geq 2^{|I|}$ .*

*Proof.* Define  $N = \mathbb{N} \setminus \{0\}$ . For every  $n \in N$ , let  $F_n$  be the free group in the variety  $\mathcal{V}$  with  $n$  generators. Define  $H = \prod_{n \in N} F_n$ , and note that  $r_{\mathcal{V}}(H) \geq \omega$ . Since  $I$  is infinite, there exists a bijection  $\xi : I \times N \rightarrow I$ . For  $(i, n) \in I \times N$ , fix a subgroup  $F_{in}$  of  $H_{\xi(i, n)}$  isomorphic to  $F_n$  (this can be done because  $r_{\mathcal{V}}(H_{\xi(i, n)}) \geq \omega$ ). Then  $\prod_{(i, n) \in I \times N} F_{in}$  is a subgroup of the group  $\prod_{(i, n) \in I \times N} H_{\xi(i, n)} \cong \prod_{i \in I} H_i$ , where  $\cong$  denotes the isomorphism between groups. Clearly,

$$\prod_{(i, n) \in I \times N} F_{in} \cong \prod_{i \in I} \prod_{n \in N} F_{in} \cong \prod_{i \in I} \prod_{n \in N} F_n \cong \prod_{i \in I} H \cong H^I,$$

so there exists a monomorphism  $f : H^I \rightarrow \prod_{i \in I} H_i$ . Now

$$r_{\mathcal{V}} \left( \prod_{i \in I} H_i \right) \geq r_{\mathcal{V}} (f(H^I)) = r_{\mathcal{V}} (H^I) \geq 2^{|I|},$$

where the the first inequality follows from [13, Corollary 2.5] and the last inequality has been proved in [13, Lemma 4.1].  $\square$

**Lemma 5.2.** *Suppose that  $I$  is an infinite set and  $H_i$  is a separable metric space for every  $i \in I$ . If  $\mathbf{Ps}(\kappa, |I|)$  holds, then the product  $H = \prod_{i \in I} H_i$  contains a  $G_{\delta}$ -dense subset of size at most  $\kappa$ .*

*Proof.* Let  $i \in I$ . Since  $H_i$  is a separable metric space,  $|H_i| \leq \mathfrak{c}$ , and so we can fix a surjection  $f_i : \mathbb{R} \rightarrow H_i$ .

Let  $\theta : \mathbb{R}^I \rightarrow H$  be the map defined by  $\theta(g) = \{f_i(g(i))\}_{i \in I} \in H$  for every  $g \in \mathbb{R}^I$ . Since  $\mathbf{Ps}(\kappa, |I|)$  holds, [13, Lemma 2.9] allows us to conclude that  $\mathbb{R}^I$  contains an  $\omega$ -dense subset  $X$  of size  $\kappa$ . Define  $Y = \theta(X)$ . Then  $|Y| \leq |X| = \kappa$ . It remains only to show that  $Y$  is  $G_{\delta}$ -dense in  $H$ . Indeed, let  $E$  be a non-empty  $G_{\delta}$ -subset of  $H$ . Then there exist a countable subset  $J$  of  $I$  and  $h \in \prod_{j \in J} H_j$  such that  $\{h\} \times \prod_{i \in I \setminus J} H_i \subseteq E$ . For every  $j \in J$  select  $r_j \in \mathbb{R}$  such that  $f_j(r_j) = h(j)$ . Since  $X$  is  $\omega$ -dense in  $\mathbb{R}^I$ , there exists  $x \in X$  such that  $x(j) = r_j$  for every  $j \in J$ . Now

$$\begin{aligned} \theta(x) &= \{f_i(x(i))\}_{i \in I} = \{f_j(x(j))\}_{j \in J} \times \{f_i(x(i))\}_{i \in I \setminus J} \\ &= \{h(j)\}_{j \in J} \times \{f_i(x(i))\}_{i \in I \setminus J} \in \{h\} \times \prod_{i \in I \setminus J} H_i \subseteq E. \end{aligned}$$

Therefore,  $\theta(x) \in Y \cap E \neq \emptyset$ .  $\square$

**Lemma 5.3.** *Let  $\kappa \geq \omega_1$  be a cardinal and  $G$  and  $H$  be topological groups in a variety  $\mathcal{V}$  such that:*

- (a)  $r_{\mathcal{V}}(H) \geq \kappa$ ,
- (b)  $H^{\omega}$  has a  $G_{\delta}$ -dense subset of size at most  $\kappa$ ,
- (c)  $G$  has a  $G_{\delta}$ -dense subset of size at most  $\kappa$ .

*Then  $G \times H^{\omega_1}$  contains a  $G_{\delta}$ -dense  $\mathcal{V}$ -independent subset of size  $\kappa$ .*

*Proof.* Since  $\kappa \geq \omega_1$ , we have  $|\kappa \times \omega_1| = \kappa$ , and so we can use item (a) to fix a faithfully indexed  $\mathcal{V}$ -independent subset  $X = \{x_{\alpha\beta} : \alpha \in \kappa, \beta \in \omega_1\}$  of  $H$ . For every  $\beta \in \omega_1 \setminus \omega$  the topological groups  $G \times H^{\omega}$  and  $G \times H^{\beta}$  are isomorphic, so we can use items (b) and (c) to fix  $\{g_{\alpha\beta} : \alpha \in \kappa\} \subseteq G$  and  $\{y_{\alpha\beta} : \alpha \in \kappa\} \subseteq H^{\beta}$  such that  $Y_{\beta} = \{(g_{\alpha\beta}, y_{\alpha\beta}) : \alpha \in \kappa\}$  is a  $G_{\delta}$ -dense subset of  $G \times H^{\beta}$ .

For  $\alpha \in \kappa$  and  $\beta \in \omega_1 \setminus \omega$  define  $z_{\alpha\beta} \in H^{\omega_1}$  by

$$(2) \quad z_{\alpha\beta}(\gamma) = \begin{cases} y_{\alpha\beta}(\gamma), & \text{for } \gamma \in \beta \\ x_{\alpha\beta}, & \text{for } \gamma \in \omega_1 \setminus \beta \end{cases} \quad \text{for } \gamma \in \omega_1.$$

Finally, define

$$Z = \{(g_{\alpha\beta}, z_{\alpha\beta}) : \alpha \in \kappa, \beta \in \omega_1 \setminus \omega\} \subseteq G \times H^{\omega_1}.$$

**Claim 5.4.** *Z is  $G_\delta$ -dense in  $G \times H^{\omega_1}$ .*

*Proof.* Let  $E$  be a non-empty  $G_\delta$ -subset of  $G \times H^{\omega_1}$ . Then there exist  $\beta \in \omega_1 \setminus \omega$  and a non-empty  $G_\delta$ -subset  $E'$  of  $G \times H^\beta$  such that

$$(3) \quad E' \times H^{\omega_1 \setminus \beta} \subseteq E.$$

Since  $Y_\beta$  is  $G_\delta$ -dense in  $G \times H^\beta$ , there exists  $\alpha \in \kappa$  such that  $(g_{\alpha\beta}, y_{\alpha\beta}) \in E'$ . From (2) it follows that  $z_{\alpha\beta} \upharpoonright \beta = y_{\alpha\beta}$ . Combining this with (3), we conclude that  $(g_{\alpha\beta}, z_{\alpha\beta}) \in E$ . Thus  $(g_{\alpha\beta}, z_{\alpha\beta}) \in E \cap Z \neq \emptyset$ .  $\square$

**Claim 5.5.** *Z is  $\mathcal{V}$ -independent.*

*Proof.* Let  $F$  be a non-empty finite subset of  $\kappa \times (\omega_1 \setminus \omega)$ . Define

$$(4) \quad \gamma = \max\{\beta \in \omega_1 \setminus \omega : \exists \alpha \in \kappa \ (\alpha, \beta) \in F\}.$$

From (2) and (4) it follows that  $z_{\alpha\beta}(\gamma) = x_{\alpha\beta}$  for all  $(\alpha, \beta) \in F$ . Therefore,

$$X_F = \{z_{\alpha\beta}(\gamma) : (\alpha, \beta) \in F\} = \{x_{\alpha\beta} : (\alpha, \beta) \in F\} \subseteq X.$$

Since  $X$  is a  $\mathcal{V}$ -independent subset of  $H$ , so is  $X_F$  [13, Lemma 2.3]. Let  $f : G \times H^{\omega_1} \rightarrow H$  be the projection homomorphism defined by  $f(g, h) = h(\gamma)$  for  $(g, h) \in G \times H^{\omega_1}$ . Define

$$S_F = \{(g_{\alpha\beta}, z_{\alpha\beta}) : (\alpha, \beta) \in F\}.$$

Since  $G \in \mathcal{V}$ ,  $H \in \mathcal{V}$ ,  $\langle S_F \rangle$  is a subgroup of  $G \times H^{\omega_1}$  and  $\mathcal{V}$  is a variety,  $\langle S_F \rangle \in \mathcal{V}$ . Since  $f \upharpoonright_{S_F} : S_F \rightarrow H$  is an injection and  $f(S_F) = X_F$  is a  $\mathcal{V}$ -independent subset of  $H$ , from [13, Lemma 2.4] we obtain that  $S_F$  is  $\mathcal{V}$ -independent. Since  $F$  was taken arbitrary, from [13, Lemma 2.3] it follows that  $Z$  is  $\mathcal{V}$ -independent.  $\square$

From the last claim we conclude that  $|Z| = |\kappa \times (\omega_1 \setminus \omega)| = \kappa$ .  $\square$

**Lemma 5.6.** *Assume that  $\kappa$  is a cardinal,  $\{H_n : n \in \mathbb{N}\}$  is a family of separable metric groups in a variety  $\mathcal{V}$  and  $\{\sigma_n : n \in \mathbb{N}\}$  is a sequence of cardinals such that:*

- (i)  $r_{\mathcal{V}}(H_n) \geq \omega$  for every  $n \in \mathbb{N}$ ,
- (ii)  $\sigma = \sup\{\sigma_n : n \in \mathbb{N}\} \geq \omega_1$ ,
- (iii)  $\mathbf{Ps}(\kappa, \sigma)$  holds.

Then  $\prod_{n \in \mathbb{N}} H_n^{\sigma_n}$  has a  $G_\delta$ -dense  $\mathcal{V}$ -independent subset of size  $\kappa$ .

*Proof.* Define

$$S = \{n \in \mathbb{N} : \sigma_n \geq \omega_1\}, \quad G = \prod_{n \in \mathbb{N} \setminus S} H_n^{\sigma_n} \quad \text{and} \quad H = \prod_{n \in S} H_n^{\sigma_n}.$$

From items (i) and (ii) of our lemma it follows that

$$(5) \quad H \cong \prod_{i \in I} H'_i, \quad \text{where } |I| = \sigma \text{ and each } H'_i \text{ is a separable metric group}$$

satisfying  $r_{\mathcal{V}}(H'_i) \geq \omega$ ,

where  $\cong$  denotes the isomorphism between topological groups. Since  $|\sigma_n \times \omega_1| = \sigma_n$  for every  $n \in S$ , we have

$$H^{\omega_1} \cong \prod_{n \in S} (H_n^{\sigma_n})^{\omega_1} \cong \prod_{n \in S} H_n^{\sigma_n \times \omega_1} \cong \prod_{n \in S} H_n^{\sigma_n} \cong H.$$

In particular,

$$\prod_{n \in \mathbb{N}} H_n^{\sigma_n} = G \times H \cong G \times H^{\omega_1}.$$

Therefore, the conclusion of our lemma would follow from that of Lemma 5.3 so long as we prove that  $G$  and  $H$  satisfy the assumptions of Lemma 5.3. From (ii), (iii) and Proposition 4.6(b) one concludes that  $\kappa \geq \mathfrak{c} \geq \omega_1$ .

Let us check that the assumption of item (a) of Lemma 5.3 holds. From (5) and Lemma 5.1 we get  $r_{\mathcal{V}}(H) \geq 2^\sigma$ . Since  $\mathbf{Ps}(\kappa, \sigma)$  holds by item (iii), we have  $2^\sigma \geq \kappa$  by Proposition 4.6(c). This shows that  $r_{\mathcal{V}}(H) \geq \kappa$ .

Let us check that the assumption of item (b) of Lemma 5.3 holds. Recalling (5), we conclude that

$$H^\omega \cong \prod_{i \in I} (H'_i)^\omega, \text{ where each } (H'_i)^\omega \text{ is a separable metric space.}$$

Since  $|I| = \sigma$  by (5), and  $\mathbf{Ps}(\kappa, \sigma)$  holds by item (iii), Lemma 5.2 allows us to conclude that  $H^\omega$  has  $G_\delta$ -dense subset of size at most  $\kappa$ .

Let us check that the assumption of item (c) of Lemma 5.3 holds. Since  $\sigma_n \leq \omega$  for every  $n \in \mathbb{N} \setminus S$ ,  $G$  is a separable metric group, and so  $|G| \leq \mathfrak{c}$ . Since  $\mathbf{Ps}(\kappa, \sigma)$  holds,  $\mathfrak{c} \leq \kappa$  by Proposition 4.6(b), and so  $G$  itself is a  $G_\delta$ -dense subset of  $G$  of size at most  $\kappa$ .  $\square$

**Corollary 5.7.** *Let  $\mathbb{P}$  be the set of prime numbers and  $\{\sigma_p : p \in \mathbb{P}\}$  a sequence of cardinals such that  $\sigma = \sup\{\sigma_p : p \in \mathbb{P}\} \geq \omega_1$ . If  $\kappa$  is a cardinal such that  $\mathbf{Ps}(\kappa, \sigma)$  holds, then the group*

$$(6) \quad K = \prod_{p \in \mathbb{P}} \mathbb{Z}_p^{\sigma_p}$$

*contains a  $G_\delta$ -dense free subgroup  $F$  such that  $|F| = \kappa$ .*

*Proof.* Since  $r(\mathbb{Z}_p) \geq \omega$  for every  $p \in \mathbb{P}$ , applying Lemma 5.6 with  $\mathcal{V} = \mathcal{A}$  we can find a  $G_\delta$ -dense  $\mathcal{A}$ -independent subset  $X$  of  $K$  of size  $\kappa$ . Since  $\mathcal{A}$ -independence coincides with the usual independence for abelian groups, the subgroup  $F$  of  $K$  generated by  $X$  is free. Clearly,  $|F| = \kappa$ . Since  $X \subseteq F \subseteq K$  and  $X$  is  $G_\delta$ -dense in  $K$ , so is  $F$ .  $\square$

As an application, we obtain the following particular case of [13, Lemma 4.3].

**Corollary 5.8.** *Let  $\kappa$  and  $\sigma \geq \omega_1$  be cardinals such that  $\mathbf{Ps}(\kappa, \sigma)$  holds. Then for every compact metric non-torsion abelian group  $H$  the group  $H^\sigma$  contains a  $G_\delta$ -dense free subgroup  $F$  such that  $|F| = \kappa$ .*

*Proof.* Since  $H$  is a compact non-torsion abelian group,  $r(H) \geq \omega$ . Applying Lemma 5.6 with  $\mathcal{V} = \mathcal{A}$ ,  $\sigma_n = \sigma$  and  $H_n = H$  for every  $n \in \mathbb{N}$ , we can find a  $G_\delta$ -dense independent subset  $X$  of  $K = H^\sigma$  of size  $\kappa$ . Then the subgroup  $F$  of  $K$  generated by  $X$  is free and satisfies  $|F| = \kappa$ . Since  $X \subseteq F \subseteq K$  and  $X$  is  $G_\delta$ -dense in  $K$ , so is  $F$ .  $\square$

## 6. ESSENTIAL FREE SUBGROUPS OF COMPACT TORSION-FREE ABELIAN GROUPS

**Lemma 6.1.** *Let  $K$  be a torsion-free abelian group and let  $F$  be a free subgroup of  $K$ . Then there exists a free subgroup  $F_0$  of  $K$  containing  $F$  as a direct summand, such that:*

- (a)  $F_0$  non-trivially meets every non-zero subgroup of  $K$ , and
- (b)  $|F_0| = |K|$ .

*Proof.* Let  $A := K/F$  and let  $\pi : K \rightarrow A$  be the canonical projection. Let  $F_2$  be a free subgroup of  $A$  with generators  $\{g_i\}_{i \in I}$  such that  $A/F_2$  is torsion. Since  $\pi$  is surjective, for every  $i \in I$  there exists  $f_i \in K$ , such that  $\pi(f_i) = g_i$ . Consider the subgroup  $F_1$  of  $K$  generated by  $\{f_i : i \in I\}$ . As  $\pi(F_1) = F_2$  is free, we conclude that  $F_1 \cap F = \{0\}$ , so  $\pi|_{F_1} : F_1 \rightarrow F_2$  is an isomorphism. Let us see that the subgroup  $F_0 = F + F_1 = F \oplus F_1$  has the required properties. Indeed, it is free as  $F_1 \cap F = \{0\}$  and both  $F, F_1$  are free. Moreover,  $K/F_0 \cong A/F_2$  is torsion and  $F$  is a direct summand of  $F_0$ . As  $K/F_0$  is torsion,

$F_0$  non-trivially meets every non-zero subgroup of  $K$ , so (a) holds true. Since  $K$  is torsion-free, (b) easily follows from (a).  $\square$

**Lemma 6.2.** *Let  $K$  be a compact torsion-free abelian group and let  $F$  be a free subgroup of  $K$ . Then there exists a free essential subgroup  $F_0$  of  $K$  with  $|F_0| = |K|$ , containing  $F$  as a direct summand.*

*Proof.* Apply Lemma 6.1.  $\square$

**Lemma 6.3.** *Suppose  $\text{Min}(\kappa, \sigma)$  holds, and let  $\{\sigma_p : p \in \mathbb{P}\}$  be the sequence of cardinals witnessing  $\text{Min}(\kappa, \sigma)$ . Let  $F$  be a free subgroup of the group  $K$  as in (6) with  $|F| = \kappa$ . Then there exists a free essential subgroup  $F'$  of  $K$  containing  $F$  as a direct summand such that  $|F'| = \kappa$ .*

*Proof.* Let

$$(7) \quad \text{wtd}(K) = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p^{\sigma_p} \text{ and } F_* = F \cap \text{wtd}(K).$$

Then  $F_*$  is a free subgroup of  $\text{wtd}(K)$ , so applying Lemma 6.1 to the group  $\text{wtd}(K)$  and its subgroup  $F_*$  we get a free subgroup  $F^*$  of  $\text{wtd}(K)$  such that:

- (i)  $F^* \supseteq F_*$  and  $F^* = F_* \oplus L$  for an appropriate subgroup  $L$  of  $F^*$ ;
- (ii)  $F^*$  non-trivially meets every non-zero subgroup of  $\text{wtd}(K)$ ;
- (iii)  $|F^*| = |\text{wtd}(K)| \leq \kappa = |F|$ .

Obviously, (ii) yields that  $F^*$  is essential in  $\text{wtd}(K)$ . As  $\text{wtd}(K)$  is essential in  $K$  [12], we conclude that  $F^*$  is essential in  $K$  as well. From (iii) we conclude that  $F' = F + F^*$  is an essential subgroup of  $K$  of size  $\kappa$  containing  $F$ . Finally, from (7) and (i) we get  $F' = F + L$ , and since  $L \subseteq \text{wtd}(K)$ , we have

$$F \cap L = F \cap \text{wtd}(K) \cap L = F_* \cap L = \{0\}.$$

Therefore,  $F' = F \oplus L$  is free.  $\square$

**Lemma 6.4.** *Let  $\kappa$  and  $\sigma \geq \omega_1$  be cardinals such that both  $\text{Min}(\kappa, \sigma)$  and  $\text{Ps}(\kappa, \sigma)$  hold. Then  $F_\kappa$  admits a zero-dimensional minimal pseudocompact group topology of weight  $\sigma$ .*

*Proof.* Let  $\{\sigma_p : p \in \mathbb{P}\}$  be a sequence of cardinals witnessing  $\text{Min}(\kappa, \sigma)$ . In particular,  $\sigma = \sup\{\sigma_p : p \in \mathbb{P}\}$ . Then the group  $K$  as in (6) is compact and zero-dimensional. Since  $\sigma \geq \omega_1$  and  $\text{Ps}(\kappa, \sigma)$  holds, by Corollary 5.7 there exists a  $G_\delta$ -dense free subgroup  $F$  of  $K$  with  $|F| = \kappa$ . Since  $\text{Min}(\kappa, \sigma)$  holds, according to Lemma 6.3 there exists a free essential subgroup  $F'$  of  $K$  containing  $F$  with  $|F'| = \kappa$ . Obviously  $F'$  is also  $G_\delta$ -dense. By Theorem 4.1  $F'$  is pseudocompact. On the other hand, by the essentiality of  $F'$  in  $K$  and Theorem 1.7, the subgroup  $F'$  of  $K$  is also minimal. Being a subgroup of the zero-dimensional group  $K$ , the group  $F'$  is zero-dimensional. Since  $F'$  is dense in  $K$ , from (6) and (1) we have  $w(F') = w(K) = \sup\{\sigma_p : p \in \mathbb{P}\} = \sigma$ . Since  $F' \cong F_\kappa$ , the subspace topology induced on  $F'$  from  $K$  will do the job.  $\square$

**Lemma 6.5.** *Let  $\kappa$  and  $\sigma \geq \omega_1$  be cardinals such that  $\kappa = 2^\sigma$ . Then  $F_\kappa$  admits a connected minimal pseudocompact group topology of weight  $\sigma$ .*

*Proof.* The group  $K = \widehat{\mathbb{Q}}^\sigma$  is compact and connected. Since  $\kappa = 2^\sigma$ ,  $\text{Ps}(\kappa, \sigma)$  holds by Proposition 4.6(d). By Corollary 5.8 there exists a  $G_\delta$ -dense free subgroup  $F$  of  $K$  with  $|F| = \kappa$ . According to Lemma 6.2 there exists a free essential subgroup  $F'$  of  $K$  containing  $F$  with  $|F'| = |K| = \kappa$ . Obviously  $F'$  is also  $G_\delta$ -dense. By Theorem 4.1  $F'$  is pseudocompact. On the other hand, by the essentiality of  $F'$  in  $K$  and Theorem 1.7, the subgroup  $F'$  of  $K$  is also minimal. Since  $G_\delta$ -dense subgroups of compact connected

abelian groups are connected [13, Fact 2.10(ii)], we conclude that  $F'$  is connected. Since  $F'$  is dense in  $K$ , we have  $w(F') = w(K) = \sigma$ . Clearly,  $F' \cong F_\kappa$  as  $|F'| = |F| = 2^\sigma = \kappa$ . Therefore, the subspace topology induced on  $F'$  from  $K$  will do the job.  $\square$

## 7. PROOFS OF THE THEOREMS FROM SECTION 2

**Lemma 7.1.** *Let  $G$  be a minimal torsion-free abelian group and  $K$  its completion. Then:*

- (i)  *$K$  is a compact torsion-free abelian group;*
- (ii) *there exists a sequence of cardinals  $\{\sigma_p : p \in \mathbb{P} \cup \{0\}\}$  such that*

$$(8) \quad K = \widehat{\mathbb{Q}}^{\sigma_0} \times \prod_{p \in \mathbb{P}} \mathbb{Z}_p^{\sigma_p}.$$

*Proof.* (i) By the precompactness theorem of Prodanov and Stoyanov ([12, Theorem 2.7.7]),  $G$  is precompact, and so  $K$  is compact. Let us show that  $K$  is torsion-free. Let  $x \in K \setminus \{0\}$ . Assume that the cyclic group  $Z = \langle x \rangle$  generated by  $x$  is finite. Then  $Z$  is closed in  $K$  and non-trivial. Since  $G$  is essential in  $K$  by Theorem 1.7, it follows that  $Z \cap G \neq \{0\}$ . Choose  $y \in Z \cap G \neq \{0\}$ . Since  $Z$  is finite,  $y$  must be a torsion element, in contradiction with the fact that  $G$  is torsion-free.

(ii) Since  $K$  is torsion-free by item (i), the Pontryagin dual of  $K$  is divisible. Now the conclusion of item (ii) of our lemma follows from [19, Theorem 25.8].  $\square$

**Proof of Theorem 2.1.** Let  $K$  be the compact completion of  $G$ . Let  $\sigma = w(K) = w(G)$ . Then clearly

$$(9) \quad |G| \leq |K| = 2^\sigma.$$

If  $\sigma = \omega$ , then  $|G| \leq |K| = 2^\sigma = \mathfrak{c}$ . Hence  $\mathbf{Min}(|G|, \sigma)$  holds according to Example 3.1. Therefore, we assume  $\sigma > \omega$  for the rest of the proof.

We consider first the case when  $G$  is torsion-free. Although this part of the proof is not used in the second part covering the general case, we prefer to include it because this provides a self-contained proof of Theorems 2.10, 2.11 and 2.12 which concern only free (hence, torsion-free) groups. Let  $\{\sigma_p : p \in \mathbb{P} \cup \{0\}\}$  be the sequence from the conclusion of Lemma 7.1(ii). Clearly, our assumption  $\sigma > \omega$  implies that  $\sigma_p > \omega$  for some  $p \in \mathbb{P} \cup \{0\}$ . Hence  $\sigma = \sup\{\sigma_p : p \in \mathbb{P} \cup \{0\}\}$ . Since  $G$  is both dense and essential in  $K$ , from [1, Theorems 3.12 and 3.14] we get

$$\sup_{p \in \{0\} \cup \mathbb{P}} 2^{\sigma_p} \leq |G|.$$

Therefore  $\mathbf{Min}(|G|, \sigma)$  holds in view of (9). Since  $\sigma = w(G)$ , we are done.

In the general case, we consider the connected component  $c(K)$  of  $K$  and the totally disconnected quotient  $K/c(K)$ . Then

$$K/c(K) \cong \prod_{p \in \mathbb{P}} K_p,$$

where each  $K_p$  is a pro- $p$ -group. Let  $\sigma_p = w(K_p)$  and  $\sigma_0 = w(c(K))$ . Our assumption  $\sigma > \omega$  implies that  $\sigma_p > \omega$  for some  $p \in \mathbb{P} \cup \{0\}$ , so that

$$\sigma = w(G) = w(K) = \sup_{p \in \{0\} \cup \mathbb{P}} \sigma_p.$$

By [1, Theorems 3.12 and 3.14], one has

$$|c(K)| \cdot \sup_{p \in \mathbb{P}} 2^{\sigma_p} \leq |G|.$$

Therefore,

$$\sup_{p \in \{0\} \cup \mathbb{P}} 2^{\sigma_p} \leq |G| \leq |K| = 2^\sigma$$

in view of (9). Thus  $\mathbf{Min}(|G|, \sigma)$  holds. Since  $\sigma = w(G)$ , we are done.  $\square$

**Proof of Theorem 2.2.** Let  $G$  be a minimal abelian group with  $w(G) \geq \kappa$ . Define  $\sigma = w(G)$ . Then  $\mathbf{Min}(|G|, \sigma)$  holds by Theorem 2.1. Let  $\{\sigma_n : n \in \mathbb{N}\}$  be a sequence of cardinals witnessing  $\mathbf{Min}(|G|, \sigma)$ . That is,

$$(10) \quad \sigma = \sup_{n \in \mathbb{N}} \sigma_n \quad \text{and} \quad \sup_{n \in \mathbb{N}} 2^{\sigma_n} \leq |G| \leq 2^\sigma.$$

If  $\text{cf}(\sigma) > \omega$ , then  $|G| = 2^\sigma \geq 2^\kappa$  by Example 3.2(c). Assume that  $\text{cf}(\sigma) = \omega$ . If  $\sigma_n = \sigma$  for some  $n \in \mathbb{N}$ , then  $|G| = 2^\sigma \geq 2^\kappa$  by Example 3.2(b). So we may additionally assume that  $\sigma_n < \sigma$  for every  $n \in \mathbb{N}$ . Since  $\text{cf}(\kappa) > \omega = \text{cf}(\sigma)$ , our hypothesis  $\sigma \geq \kappa$  gives  $\sigma > \kappa$ . Then  $\sigma_n \geq \kappa$  for some  $n \in \mathbb{N}$ , and so  $|G| \geq 2^{\sigma_n} \geq 2^\kappa$  by (10).  $\square$

**Proof of Theorem 2.5.** By Theorem 2.1,  $\mathbf{Min}(|G|, w(G))$  holds. Since  $|G|$  is assumed to be non-exponential, the conclusion now follows from Proposition 3.5(b).  $\square$

**Proof of Theorem 2.8.** Since  $|F_\kappa| = \kappa$ , from our assumption and Theorem 2.1 we conclude that  $\mathbf{Min}(\kappa, \sigma)$  holds. Lemma 4.9 yields that  $\mathbf{Ps}(\kappa, \sigma)$  holds as well. Since  $\sigma$  is infinite and not a strong limit, it follows that  $\sigma \geq \omega_1$ . Now Lemma 6.4 applies.  $\square$

**Proof of Theorem 2.10.** The implications (c) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a) are obvious.

(a) $\Rightarrow$ (c) Assume that  $\tau_1$  is a minimal topology of weight  $\sigma$  on  $F_\kappa$ . Then  $\sigma \geq \omega_1$  as  $\kappa > \mathfrak{c}$ . According to Theorem 2.1  $\mathbf{Min}(\kappa, \sigma)$  holds. Now assume that  $\tau_2$  is a minimal topology of weight  $\lambda$  on  $F_\kappa$ . According to Theorem 4.3  $\mathbf{Ps}(\kappa, \lambda)$  holds. Now Lemma 4.11 yields that also  $\mathbf{Ps}(\kappa, \sigma)$  holds true. Finally, the application of Lemma 6.4 finishes the proof.  $\square$

**Remark 7.2.** It is clear from the above proof that the topologies from items (b) and (c) of Theorem 2.10 can be chosen to have the same weight  $\sigma$  as the minimal topology from item (a) of this theorem.

**Proof of Theorem 2.11.** The implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) are obvious.

(a) $\Rightarrow$ (d) Suppose that  $F_\mathfrak{c}$  admits a minimal pseudocompact group topology. Since  $F_\mathfrak{c}$  is free,  $F_\mathfrak{c}$  does not admit any compact group topology, and so  $\mathfrak{c} = |F_\mathfrak{c}| \geq 2^{\omega_1}$  by Corollary 2.7. The converse inequality  $\mathfrak{c} \leq 2^{\omega_1}$  is clear.

(d) $\Rightarrow$ (b) Follows from  $\mathfrak{c} = 2^{\omega_1}$  and Lemma 6.5.

(d) $\Rightarrow$ (c) Follows from  $\mathfrak{c} = 2^{\omega_1}$  and Lemma 6.4, as  $\mathbf{Min}(\mathfrak{c}, \omega_1)$  holds by Example 3.2(a), and  $\mathbf{Ps}(\mathfrak{c}, \omega_1)$  holds by Proposition 4.6(a).  $\square$

**Proof of Theorem 2.12.** (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c) Assume that  $\tau_1$  is a connected minimal group topology on  $F_\kappa$  with  $w(F_\kappa, \tau_1) = \sigma$ . Then the completion  $K$  of  $(F_\kappa, \tau_1)$  satisfies the conclusion of Lemma 7.1(ii). Moreover,  $K$  is connected. Since the zero-dimensional group

$$L = \prod_{p \in \mathbb{P}} \mathbb{Z}_p^{\sigma_p}$$

from (8) is a continuous image of the connected group  $K$ , we must have  $L = \{0\}$ . It follows that  $K = \widehat{\mathbb{Q}}^{\sigma_0}$ . Note that  $\sigma_0 = w(K) = w(F_\kappa, \tau_1) = \sigma$ . That is,  $K = \widehat{\mathbb{Q}}^\sigma$ . Since  $F_\kappa$  is both dense and essential in  $K$  by Theorem 1.7, from [1, Theorems 3.12 and 3.14] we get  $2^\sigma \leq |F_\kappa| \leq |K| = 2^\sigma$ . Hence  $\kappa = 2^\sigma$ .

(c) $\Rightarrow$ (a) Follows from  $\kappa = 2^\sigma$  and Lemma 6.5.  $\square$

**Proof of Theorem 2.13.** Let  $G$  be a locally connected minimal abelian group and  $K$  its completion. Let  $U$  be a non-empty open connected subset of  $G$ . Choose an open subset  $V$  of  $K$  such that  $V \cap G = U$ . Since  $U$  is dense in  $V$  and  $U$  is connected, so is  $V$ . Therefore,  $K$  is locally connected. Applying Lemma 7.1(i), we conclude that  $K$  is compact and torsion-free. From [13, Corollary 8.8] we get  $K = \{0\}$ . Hence  $G$  is trivial as well.  $\square$

## 8. FINAL REMARKS AND OPEN QUESTIONS

The divisible abelian groups that admit a minimal group topology were described in [8]. Here we need only the part of this characterization for divisible abelian groups of size  $\geq \mathfrak{c}$ .

**Theorem 8.1.** [8] *A divisible abelian group of cardinality at least  $\mathfrak{c}$  admits some minimal group topology precisely when it admits a compact group topology.*

The concept of pseudocompactness generalizes compactness from a different angle than that of minimality. It is therefore quite surprising that minimality and pseudocompactness *combined together* “yield” compactness in the class of divisible abelian groups. This should be compared with Corollary 2.7, where a similar phenomenon (i.e., minimal and pseudocompact topologizations imply compact topologization) occurs for all “small” groups.

The next theorem shows that the counterpart of the simultaneous minimal and pseudocompact topologization of divisible abelian groups is much easier than that of free abelian groups.

**Theorem 8.2.** *A divisible abelian group admits a minimal group topology and a pseudocompact group topology if and only it admits a compact group topology.*

*Proof.* The necessity is obvious. To prove the sufficiency, suppose that a divisible abelian group  $G$  admits both a minimal group topology and a pseudocompact group topology. If  $G$  is finite, then  $G$  admits a compact group topology. If  $G$  is infinite, then  $|G| \geq \mathfrak{c}$  by Theorem 1.8. Now the conclusion follows from Theorem 8.1.  $\square$

Our next example demonstrates that both the restriction on the cardinality in Theorem 8.1 and the hypothesis of the existence of a pseudocompact group topology in Theorem 8.2 are needed:

**Example 8.3.** (a) The divisible abelian group  $\mathbb{Q}/\mathbb{Z}$  admits a minimal group topology [10], but does not admit a pseudocompact group topology (Theorem 1.8).  
 (b) The divisible abelian group  $\mathbb{Q}^{(\mathfrak{c})} \oplus (\mathbb{Q}/\mathbb{Z})^{(\omega)}$  admits a (connected) pseudocompact group topology [13], but does not admit any minimal group topology. The latter conclusion follows from Theorem 8.1 and the fact that this group does not admit any compact group topology [19].

Let us briefly discuss the possibilities to extend our results for free abelian groups to the case of torsion-free abelian groups. Theorem 8.2 shows that for divisible torsion-free abelian groups the situation is in some sense similar to that of free abelian groups described in Theorem 2.10: in both cases the existence of a pseudocompact group topology and a minimal group topology is equivalent to the existence of a minimal pseudocompact (actually, compact) group topology. Nevertheless, there is a substantial difference, because free abelian groups admit no compact group topology. Another important difference between both cases is that Problem 1.2 is still open for torsion-free abelian groups [9]:

**Problem 8.4.** *Characterize the minimal torsion-free abelian groups.*

A quotient of a minimal group need not be minimal even in the abelian case. This justified the isolation in [10] of the smaller class of totally minimal groups:

**Definition 8.5.** A topological group  $G$  is called *totally minimal* if every Hausdorff quotient group of  $G$  is minimal. Equivalently, a Hausdorff topological group  $G$  is totally minimal if every continuous group homomorphism  $f : G \rightarrow H$  of  $G$  onto a Hausdorff topological group  $H$  is open.

It is clear that compact  $\Rightarrow$  totally minimal  $\Rightarrow$  minimal. Therefore, Theorem 2.10 makes it natural to ask the following question:

**Question 8.6.** Let  $\kappa > \mathfrak{c}$  be a cardinal.

- (a) When does  $F_\kappa$  admit a totally minimal group topology?
- (b) When does  $F_\kappa$  admit a totally minimal pseudocompact group topology?

More specifically, one can ask:

**Question 8.7.** Let  $\kappa > \mathfrak{c}$  be a cardinal. Is the condition “ $F_\kappa$  admits a zero-dimensional totally minimal pseudocompact group topology” equivalent to those of Theorem 2.10?

Since  $F_\mathfrak{c}$  admits a totally minimal group topology [21] and a pseudocompact group topology [13], the obvious counter-part of Theorem 2.11 suggests itself:

**Question 8.8.** Assume the Lusin’s Hypothesis  $2^{\omega_1} = \mathfrak{c}$ .

- (i) Does  $F_\mathfrak{c}$  admit a totally minimal pseudocompact group topology?
- (ii) Does  $F_\mathfrak{c}$  admit a totally minimal pseudocompact connected group topology?
- (iii) Does  $F_\mathfrak{c}$  admit a totally minimal pseudocompact zero-dimensional group topology?

Let us mention another class of abelian groups where both problems (Problem 1.2 for minimal group topologies [11] and its counterpart for pseudocompact group topologies [6, 13]) are completely resolved. These are the torsion abelian groups. Nevertheless, we do not know the answer of the following question:

**Question 8.9.** Let  $G$  be a torsion abelian group that admits a minimal group topology and a pseudocompact group topology. Does  $G$  admit also a minimal pseudocompact group topology?

We finish with the question about (non-abelian) free groups. We note that the topology from Theorem 1.6 is even totally minimal. Furthermore, a free group  $F$  admits a pseudocompact group topology if and only if  $\mathbf{Ps}(|F|, \sigma)$  holds for some infinite cardinal  $\sigma$  [13]. This justifies our final

**Question 8.10.** Let  $F$  be a free group that admits a pseudocompact group topology.

- (i) Does  $F$  have a minimal pseudocompact group topology?
- (ii) Does  $F$  have a totally minimal pseudocompact group topology?
- (iii) Does  $F$  have a (totally) minimal pseudocompact connected group topology?
- (iv) Does  $F$  have a (totally) minimal pseudocompact zero-dimensional group topology?

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